UNIT – III   ~~Sec~~ Chapter – 3

Series of real numbers.

The Series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots \infty$

Defn 3.1A The infinite series $\sum_{n=1}^{\infty} a_n$ is an ordered pair $\langle \{a_n\}_{n=1}^{\infty}, \{s_n\}_{n=1}^{\infty} \rangle$ where $\{a_n\}_{n=1}^{\infty}$ is a Sequence of real numbers and $s_n = a_1 + a_2 + \cdots + a_n$ $(n \in I)$

The number a_n is called the n^{th} term of the series, The number s_n is called the n^{th} partial sum of the series.

Defn 3.1B Convergent and divergent of the series.

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers with partial sums $s_n = a_1 + a_2 + \cdots + a_n$ $(n \in I)$.

If the sequence $\{s_n\}_{n=1}^{\infty}$ converges to $A \in R$, we Say that the series $\sum_{n=1}^{\infty} a_n$ converges to A.

If $\{s_n\}_{n=1}^{\infty}$ diverges, we say that $\sum_{n=1}^{\infty} a_n$ diverges.

If $\sum_{n=1}^{\infty} a_n$ converges to A, we can write $\sum_{n=1}^{\infty} a_n = A$

If $\sum_{n=1}^{\infty} a_n$ diverges to ∞ we can write $\sum_{n=1}^{\infty} a_n = \infty$.

Theorem 3.1 C: If $\sum_{n=1}^{\infty} a_n$ converges to A and $\sum_{n=1}^{\infty} b_n$ converges to B, then $\sum_{n=1}^{\infty}(a_n+b_n)$ converges to A+B. Also if $c \in R$, then $\sum_{n=1}^{\infty} ca_n$ converges to cA.

Proof: If $s_n = a_1+a_2+\cdots+a_n$ and $t_n = b_1+b_2+\cdots+b_n$

given $\sum_{n=1}^{\infty} a_n$ converges to A

$\Rightarrow$ sequence $\{s_n\}_{n=1}^{\infty}$ converges to A

$$\lim_{n\to\infty} s_n = A \quad \text{——} \; (1)$$

Also given $\sum_{n=1}^{\infty} b_n$ converges to B

$\Leftrightarrow$ $\{t_n\}_{n=1}^{\infty}$ converges to B

$$\lim_{n\to\infty} t_n = B \quad \text{——} \; (2)$$

using (1) & (2)

$$\Rightarrow \lim_{n\to\infty}(s_n+t_n) = A+B \quad \text{——} \; (3)$$

$\Rightarrow$ ~~$\lim_{n\to\infty}$~~ But the n^{th} partial sum of $\sum_{n=1}^{\infty}(a_n+b_n)$ is

$(a_1+b_1)+(a_2+b_2)+\cdots+(a_n+b_n) = s_n+t_n$

using (3) The seq of partial sum $\{s_n+t_n\}_{n=1}^{\infty}$ converges to A+B

$\therefore$ The series $\sum_{n=1}^{\infty}(a_n+b_n)$ converges to A+B eqn ① $\lim_{n\to\infty} s_n = A$

$\therefore \lim_{n\to\infty} cs_n = cA$. —— ④

consider the series $\sum_{n=1}^{\infty} ca_n = ca_1 + ca_2 + \cdots + ca_n + \cdots$

$cs_n = c(a_1 + a_2 + \cdots + a_n)$

$= ca_1 + ca_2 + \cdots + ca_n$

using ④ The seq of partial $\{cs_n\}_{n=1}^{\infty}$ converges to cA

$\therefore$ The series $\sum_{n=1}^{\infty} ca_n$ converges to cA.

Theorem 3.1D: If $\sum_{n=1}^{\infty} a_n$ is a convergent series, then $\lim_{n\to\infty} a_n = 0$.

given $\sum_{n=1}^{\infty} a_n$ is a convergent series.

i.e) $\sum_{n=1}^{\infty} a_n = A$ say.

$\Rightarrow \lim_{n\to\infty} s_n = A$ where $s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$.

$\lim_{n\to\infty} s_{n-1} = A$ where $s_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$

$\Rightarrow \lim_{n\to\infty} (s_n - s_{n-1}) = A - A$

$\Rightarrow \lim_{n\to\infty} a_n = 0$

Prove that the series $\sum_{n=1}^{\infty} \left((1-n)/(1+2n) \right)$ is divergent. (4)

Here $a_n = \left(\frac{1-n}{1+2n}\right)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1-n}{1+2n}\right) = \lim_{n\to\infty}\left(\frac{n\left(\frac{1}{n}-1\right)}{n\left(\frac{1}{n}+2\right)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\frac{1}{n}-1}{\frac{1}{n}+2}\right) = -\frac{1}{2} \neq 0$$

$\lim_{n\to\infty} a_n \neq 0$ ∴ The series $\sum_{n=1}^{\infty}\left(\frac{1-n}{1+2n}\right)$ is divergent.

If $\sum_{n=1}^{\infty} a_n$ is convergent definitely $\lim_{n\to\infty} a_n = 0$.

But the converse is not true

i.e.) If $\lim_{n\to\infty} a_n = 0$ is not sufficient to ensure that $\sum_{n=1}^{\infty} a_n$ be convergent.

Ex The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent series

But $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is converges.

To prove sequence of partial sum of the series $\{S_n\}_{n=1}^{\infty}$ is convergent

Here $a_n = \frac{1}{n(n+1)}$

$$a_n = \frac{A}{n} + \frac{B}{n+1} \quad \text{——— (1)}$$

$$\frac{1}{n(n+1)} = \frac{A(n+1)+Bn}{n(n+1)}$$

$$1 = A(n+1) + Bn$$

Put $n=0$

$$\boxed{1 = A}$$

Put $n=-1$

$$1 = -B \quad \boxed{B=-1}$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

$$a_1 = \frac{1}{1} - \frac{1}{2}$$

$$a_2 = \frac{1}{2} - \frac{1}{3}$$

$$a_3 = \frac{1}{3} - \frac{1}{4}$$

$$\vdots$$

$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

$$S_n = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = 1 - \frac{1}{\infty} = 1$$

$\lim_{n\to\infty} S_n = 1$ sequence of partial sum of the series $\{S_n\}_{n=1}^{\infty}$ converges to 1 $\therefore$ The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is converges to 1.

Sec 3.2 Series with nonnegative terms.

Theorem 3.2A. If $\sum_{n=1}^{\infty} a_n$ is a Series of nonnegative numbers with $s_n = a_1 + a_2 + \cdots + a_n$ $(n \in I)$, then

(i) $\sum_{n=1}^{\infty} a_n$ converges if the sequence $\{s_n\}_{n=1}^{\infty}$ is bounded.

(ii) $\sum_{n=1}^{\infty} a_n$ diverges if $\{s_n\}_{n=1}^{\infty}$ is not bounded.

Proof Given $\sum_{n=1}^{\infty} a_n$ is a series of non negative numbers

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} + \cdots$$

$s_1 = a_1$ $s_2 = a_1 + a_2$ $s_3 = a_1 + a_2 + a_3, \ldots$

$s_n = a_1 + a_2 + \cdots + a_n$

$s_1 \le s_2 \le s_3 \le \cdots s_n \le s_{n+1} \le \cdots$

$\Rightarrow$ sequence $\{s_n\}_{n=1}^{\infty}$ increasing sequence

Also given $\{s_n\}_{n=1}^{\infty}$ is bounded $\Rightarrow \{s_n\}_{n=1}^{\infty}$ bdd below and bounded above.

$\Rightarrow \{s_n\}_{n=1}^{\infty}$ is increasing sequence and bounded above

$\Rightarrow$ seq $\{s_n\}_{n=1}^{\infty}$ is convergent.

$\Rightarrow$ The series $\sum_{n=1}^{\infty} a_n$ is convergent.

Hence proved (i)

Proof (ii) If $\{s_n\}_{n=1}^{\infty}$ is not bounded

$s_1 \le s_2 \le s_3 \le \cdots \le s_n \le s_{n+1} \le \cdots$

$\forall n$ $s_n \ge s_1$

$\Rightarrow \{s_n\}_{n=1}^{\infty}$ is bounded below $\therefore \{s_n\}_{n=1}^{\infty}$ is not bounded above

$\Rightarrow \{s_n\}_{n=1}^{\infty}$ is increasing sequence and not bounded above $\Rightarrow \{s_n\}_{n=1}^{\infty}$ is divergent

$\Rightarrow$ The series $\sum_{n=1}^{\infty} a_n$ is divergent.

3.2B Theorem (i) If $0 < x < 1$ then $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{(1-x)}$

(ii) If $x \geq 1$, then $\sum_{n=0}^{\infty} x^n$ diverges.

Let $0 < x < 1$, $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^{n-1} + x^n + \cdots$

$$s_n = 1 + x + x^2 + \cdots + x^{n-1}$$

$$s_n = \frac{1 - x^n}{(1-x)}$$

$$s_n = \frac{1}{(1-x)} - \frac{x^n}{(1-x)}$$

$$\lim_{n\to\infty} s_n = \frac{1}{(1-x)} - \lim_{n\to\infty} \frac{x^n}{(1-x)}$$

we know that If $0 < x < 1$ $\lim_{n\to\infty} x^n = 0$

$$\lim_{n\to\infty} s_n = \frac{1}{1-x}$$

$\Rightarrow$ The Series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{(1-x)}$

Proof (ii) The series $\sum_{n=0}^{\infty} x^n$

$$a_n = x^n$$

If $x \geq 1$ $\lim a_n = \lim_{n\to\infty} x^n \neq 0$ ~~If~~

$$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=0}^{\infty} x^n \text{ diverges to } \infty.$$

3.2C Theorem: The Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots, \quad S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Consider the subsequence $\{S_{2^n}\}_{n=0}^{\infty}$

$$S_{2^0} = S_1 = 1$$

$$S_{2^1} = S_2 = 1 + \tfrac{1}{2} = \frac{3}{2}$$

$$S_{2^2} = S_4 = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} > \frac{3}{2} + \frac{1}{4} + \frac{1}{4} = \frac{6+1+1}{4} = 2$$

$$S_{2^3} = S_8 = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} = > 2 + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}$$

$$S_{2^3} = S_8 > \frac{5}{2}$$

In general $S_{2^n} > \left(\frac{n+2}{2}\right)$

$\{S_{2^n}\}_{n=0}^{\infty}$ bounded below but not bounded above

But $\{S_{2^n}\}_{n=0}^{\infty}$ increasing and not bounded above

$\Rightarrow$ seq $\{S_{2^n}\}_{n=0}^{\infty}$ diverges to ∞.

But seq $\{S_{2^n}\}_{n=0}^{\infty}$ is sub sequence of $\{S_n\}_{n=0}^{\infty}$

$\Rightarrow$ The seq $\{S_n\}_{n=0}^{\infty}$ is diverges to ∞

$\therefore$ The Series $\sum^{\infty} \frac{1}{n}$ is divergent.